Logics for Sizes with Union or Intersection

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February 25, 2020

Thanks, [PC]. I'm going to talk about two cute little logics that can reason completely about size comparison alongside intersection (in the first) and union (in the second). These logics are computationally light; the best way to see them is as fragments of more complex logics (like first-order logic and Boolean Algebra with Presburger Arithmetic). Unfortunately, inference in these more expressive logics is either undecidable or intractable in general. But the logics in this talk are small enough to be polynomial-time decidable. So you can think of these logics as efficient fragments of BAPA, involving only cardinality comparison and intersections or unions.

It's going to get confusing for me to say "intersections or unions" all the time, so let's focus on the logic with intersection. The logic with union is very similar.

What kind of inference?

First, let's clear up exactly what kind of inference we're talking about. Here's a characteristic kind of inference our system can make: If all cats are mammals that purr, and there are at least as many cats as things that purr, then we can conclude that all things that purr are cats. I'm going to pause to let you all make the inference yourself.

(Pause to let the audience think this over)

This kind of inference has three basic components. First, a reasoner needs to know something about nonstrict cardinality comparison (the thing going on there with AtLeast). We also need to know something about set intersection – English phrases like "mammals that purr" suggest something like the intersection between mammals and purring things. Finally, there aren't many interesting things we can say about set sizes and intersection without also including set containment, which in this example is indicated by the word All.

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All cats are mammals that purr

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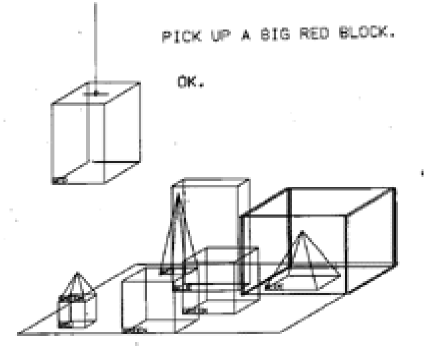
Size & Intersection Inference in AI

This kind of inference is abundant in artificially intelligent systems, especially those with explicit knowledge representation and reasoning. Since this is the section on inference and reasoning, the experts in the room probably don't need to be convinced of this. But those of you who come from learning, NLP, or otherwise, maybe it would help to set a broader context.

A fun historical example is Terry Winograd's SHRDLU, which makes basic inferences about its small BLOCKS world. Via teletype, a user could ask SHRDLU questions (in a fixed language) about the various sizes, positions, colors, and shapes of the objects. The user could also ask SHRDLU to rearrange the objects.

Because the BLOCKS world was so simple, SHRDLU was able to make fairly sophisticated inferences about its world. Lots of these inferences involved size comparison, and a couple involve intersections between properties (like "that green pyramid"). I dug through the usual SHRDLU demo and picked out the best example I could find with this kind of inference. In the third line, the user asks SHRDLU to determine whether there is a block contained in the box (which we can represent as set containment) that is narrower (here, size comparison) than "the [box] which I told you to pick up" (which, if we squint hard enough, is an intersection.

Size & Intersection Inference in AI



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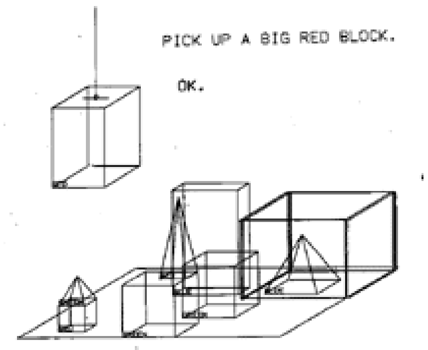


```
Person: HOW MANY BLOCKS ARE NOT IN THE BOX?
Computer: FOUR OF THEM.
Person: IS AT LEAST ONE OF THEM NARROWER THAN THE ONE WHICH I TOLD YOU TO PICK UP?
Computer: YES, THE RED CUBE.
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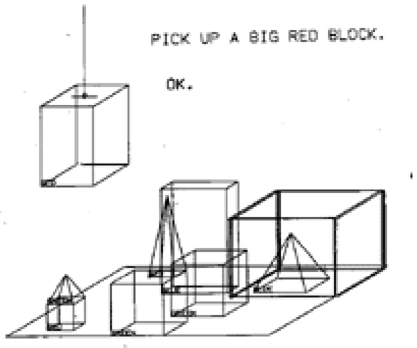


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Terry Winograd. *Procedures as a representation for data in a computer program for understanding natural language*. Tech. rep. MASSACHUSETTS INST OF TECH CAMBRIDGE PROJECT MAC, 1971

Size & Intersection Inference in AI

Most AI systems have moved beyond this kind of simple inference. But many hip-and-happening AI systems still make use of this type of reasoning.

First, Microsoft's SMT solver Z3 has recently been extended by Philippe Suter to allow for reasoning in Boolean Algebra with Presburger Arithmetic. So their plugin supports inferences on set relations (in Boolean Algebra), including set intersection and union. Their plugin also supports numerical relations (in Presburger arithmetic) and a cardinality function, which can then express cardinality comparison.

Additionally, the DeepQA framework used in IBM's Watson has a number of smaller components dealing with structured inference from knowledge bases. One of these makes use of cardinality comparison and containment when determining spatial relations between geographical regions. Two of the main spatial relations DeepQA detects are distance and region containment. Although this reasoning is not explicitly encoded logically, DeepQA does make use of certain logical properties (such as the transitivity of containment – which is an explicit rule in our logic!).

If you want to read this in more detail, I recommend taking a look at Kalyanpur's paper describing how DeepQA uses structured inference.

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A. Kalyanpur et al. "Structured data and inference in DeepQA". In: *IBM Journal of Research and Development* 56.3.4 (May 2012), 10:1–10:14. ISSN: 0018-8646. DOI: 10.1147/JRD.2012.2188737

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Syntax

So far, I've only shown you reasoning about sizes with intersection "out in the wild" (and I've been pretty generous in interpreting some of these examples to fit this kind of reasoning). So I should clarify what syntax, specifically, our logic uses.

I'll illustrate with another example inference our system can make. This inference is in fact valid, but this takes some time to see so we won't dwell on it. Let's see how we would translate this inference using formal language.

First, all noun phrases involved become base terms (I'm going to use C to stand in for "things that can vote"). I'm gonna go ahead and turn "vegetarian people" into an intersection term. We encode "there are at least as many as" using the simplified relation $\mathsf{AtLeast}$. And as for "all x are y", we have almost no simplification to do.

Alright, so this is the formal syntax of our logic. We have the two relations $\mathsf{AtLeast}$ and All, and our terms can either be base terms like H, C, V, P, or can be intersection terms like $V \cap P$.

I should mention that our system only includes nonstrict cardinality comparison. You might be wondering about strict cardinality comparison, for example "there are more apples than oranges." I'll talk more about strict cardinality comparison towards the end.

Syntax

All hippies can vote
All hippies are vegetarian
All people can vote
There are at least as many people as things that can vote
There are at least as many hippies as people

All vegetarian people are hippies

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All H *can vote*
All H *are* V
All P *can vote*
There are at least as many P *as things that can vote*
There are at least as many H *as* P

All vegetarian people are H

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Semantics for $\mathcal{A}^{\cap}$(card)

We call the logic with intersection "A-inter-card." The semantics are probably what you would expect. Basic terms are just interpreted as sets (in particular, as subsets of an underlying universe). An intersection term $a \cap b$ just denotes the intersection of the two sets a and b.

The AtLeast relation just denotes "greater than or equal" on cardinalities. So AtLeast x y holds whenever the cardinality of x is greater than or equal to the cardinality of y. And finally, the All relation is just set containment: All x y holds whenever the set x denotes is a subset of that for y.

As for entailment, φ follows from Γ if every finite model $\mathcal{M}$ that satisfies the relations in Γ also satisfies φ. Notice that we restrict the models to be finite. This means that we only consider finite sets when interpreting basic terms. This is stricter than what logicians usually mean by entailment.

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As for entailment, φ follows from Γ if every finite model $\mathcal{M}$ that satisfies the relations in Γ also satisfies φ. Notice that we restrict the models to be finite. This means that we only consider finite sets when interpreting basic terms. This is stricter than what logicians usually mean by entailment.

Semantics for $\mathcal{A}^{\cap}$(card)

$$[\![a]\!] \subseteq M$$

$$[\![a \cap b]\!] = [\![a]\!] \cap [\![b]\!]$$

$$\mathcal{M} \models \mathsf{AtLeast}\ x\ y \text{ iff } |[\![x]\!]| \geq |[\![y]\!]|$$

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Models $\mathcal{M}$ are finite!

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Inference rules for $\mathcal{A}^{\cap}(\text{card})$

$$\frac{}{\textsf{All}\ x\ x}\ (\textsc{axiom}) \qquad \frac{\textsf{All}\ x\ y \quad \textsf{All}\ y\ z}{\textsf{All}\ x\ z}\ (\textsc{barbara})$$

$$\frac{\textsf{All}\ x\ y \quad \textsf{AtLeast}\ x\ y}{\textsf{All}\ y\ x}\ (\textsc{mix}) \qquad \frac{\textsf{All}\ x\ y}{\textsf{AtLeast}\ y\ x}\ (\textsc{size}) \qquad \frac{\textsf{AtLeast}\ x\ y \quad \textsf{AtLeast}\ y\ z}{\textsf{AtLeast}\ x\ z}\ (\textsc{trans})$$

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These are the rules of inference for our logic with intersection. This turns out to be enough for completeness! Proving completeness is a bit complicated, but most of these rules are ones you might expect. (barbara) and (trans) just state transitivity of subset and AtLeast. So (barbara) says: if x is contained in y, and y is contained in z, then x must be contained in z.

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Inference rules for $\mathcal{A}^{\cup}$(card)

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$$\frac{}{\text{All } a\ (a \cup b)}\ (\text{UNION-L}) \qquad \frac{}{\text{All } b\ (a \cup b)}\ (\text{UNION-R}) \qquad \frac{\text{All } a\ c \quad \text{All } b\ c}{\text{All } (a \cup b)\ c}\ (\text{UNION-ALL})$$

Our logic with union is very similar. The only difference are those union rules highlighted in red. Here we have (union-all), which states that if two sets are contained within a bigger set c, then the union of the two is also contained in c.

Main technical results

Theorem

The logic $\mathcal{A}^{\cap}(\text{card})$ is complete.

I mentioned before that the rules on the previous slide for our logic with intersection are complete! In logics that have reductio ad absurdum as a rule, completeness would be reduced to being able to construct a model of any set Γ in our logic. But our logic does not have reductio. So in the paper, we do give a model construction, but we also had to do a little extra work to make the proof go through.

Because of the duality between the rules with intersection and the rules with union, the completeness of the union logic follows from the completeness of the intersection logic.

Maybe it's a bit surprising that these rules are complete, considering how darn simple the rules actually are. But because our rules are so simple, we also get polynomial-time decidability! The proof of this is completely syntactic, and so doesn't involve our model-building construction at all. We essentially follow an adapted version of McAllester's Tractablility Lemma (cited below): We have finitely many rules, and each rule involves a finite number of terms being substituted. This lets us polynomially bound the height of proof trees in this logic, which lets us decide whether φ follows from Γ in polynomially many steps.

I mentioned that our proof of completeness centered around model construction. I think you'll appreciate how we do it: When building a model, each of the terms a, b, c, and so on are denoted by sets. What you want to do is satisfy Γ, which prescribes the size order that terms need to have. So what we do is: line up the intersection terms (which includes base terms, since a is just $a \cap a$). Then we perform insertion sort on them, where we "swap" two terms by giving base terms the right number of elements. The proof is pretty cute, and since we mainly perform insertion sort we also get polynomial-time model building!

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David A. McAllester. "Automatic Recognition of Tractability in Inference Relations". In: *Journal of the ACM* 40 (1993), pp. 284–303

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Main technical results

Theorem
The logic $\mathcal{A}^{\cap}(\text{card})$ is complete.

Theorem
$\vdash$ is decidable in polynomial time!

Theorem
If $\Gamma \nvdash \varphi$, then we can construct a countermodel $\mathcal{M}$ satisfying Γ but falsifying φ in polynomial time

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Additional inference rules for $\mathcal{M}^{\cup}$(card)/$\mathcal{M}^{\cap}$(card)

$$\frac{\mathsf{More}\ x\ y \quad \mathsf{AtLeast}\ y\ z}{\mathsf{More}\ x\ z}\ (\textsc{more-l}) \qquad \frac{\mathsf{AtLeast}\ x\ y \quad \mathsf{More}\ y\ z}{\mathsf{More}\ x\ z}\ (\textsc{more-r})$$

$$\frac{\mathsf{More}\ x\ y}{\mathsf{AtLeast}\ x\ y}\ (\textsc{more-atleast}) \qquad \frac{\mathsf{More}\ z\ z}{\varphi}\ (\textsc{x}) \qquad \frac{\begin{matrix}[\mathsf{AtLeast}\ y\ x] \\ \vdots \\ \mathsf{More}\ z\ z\end{matrix}}{\mathsf{More}\ x\ y}\ (\textsc{raa})$$

I promised that I would get to strict cardinality comparison. In our paper, we actually do give a complete set of rules for the logic involving More. We just add the rules on this slide to the previous rules for the intersection logic.

(more-l) and (more-r) are essentially transitivity rules for strict More. (more-atleast) states that if we know that there are more x than y, we can weaken this to "there are at least as many x as y. The rule (x) is how we encode the principle of explosion in our logic – if there are strictly more things in z than in z, this is clearly a contradiction. So as usual, we can prove anything from this. Finally, we have a version of reductio ad absurdum, where if we assume that there are at least as many y as x and arrive at a contradiction, then we conclude the negation: That there are more x than y.

The problem with strict comparison is that these rules no longer enable us to decide inference in polynomial time. The issue is this reductio ad absurdum rule. Proof trees using the reductio rule could be any height whatsoever, and so we can't polynomially bound the number of steps this inference would take. We've been thinking of possible alternative rules that can replace reductio, and if you're interested I can show you one of these at my poster later this evening.

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Additional inference rules for $\mathcal{M}^{\cup}$(card)/$\mathcal{M}^{\cap}$(card)

$$\frac{\text{More } x\ y \quad \text{AtLeast } y\ z}{\text{More } x\ z}\ (\text{MORE-L}) \qquad \frac{\text{AtLeast } x\ y \quad \text{More } y\ z}{\text{More } x\ z}\ (\text{MORE-R})$$

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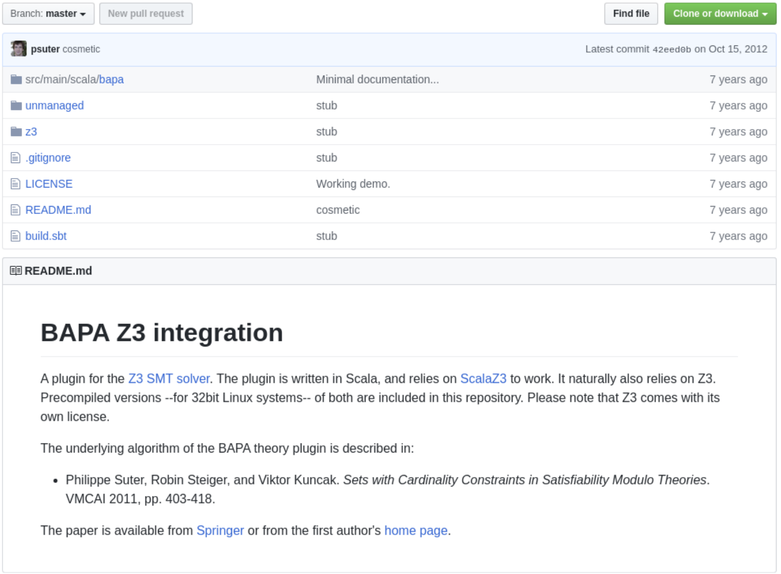
Integration with theorem provers



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psuter cosmetic — Latest commit 42eed0b on Oct 15, 2012

src/main/scala/bapa	Minimal documentation...	7 years ago
unmanaged	stub	7 years ago
z3	stub	7 years ago
.gitignore	stub	7 years ago
LICENSE	Working demo.	7 years ago
README.md	cosmetic	7 years ago
build.sbt	stub	7 years ago

README.md

BAPA Z3 integration

A plugin for the Z3 SMT solver. The plugin is written in Scala, and relies on ScalaZ3 to work. It naturally also relies on Z3. Precompiled versions --for 32bit Linux systems-- of both are included in this repository. Please note that Z3 comes with its own license.

The underlying algorithm of the BAPA theory plugin is described in:

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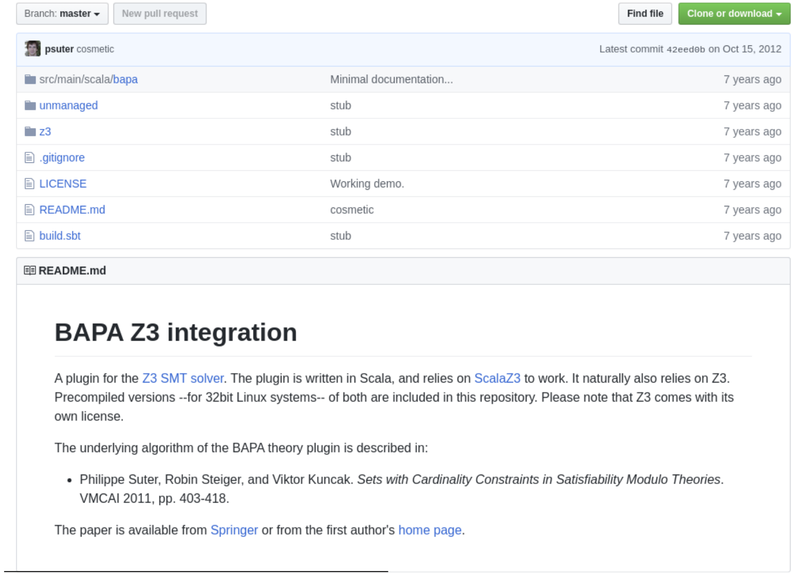
The paper is available from Springer or from the first author's home page.

Our logic is efficient, and so it would be nice to embed our logic in a theorem prover to get some speedup on this fragment of reasoning. I mentioned that Philippe Suter has extended Z3 to reason via BAPA, and so this plugin is a prime candidate to try. I'm not as familiar with building theorem provers, so I would love to talk to any of you that are interested.

Integration with theorem provers



Branch: **master** ▾ | New pull request | Find file | Clone or download ▾

psuter cosmetic — Latest commit 42eed0b on Oct 15, 2012

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Philippe Suter. *Solving Boolean Algebra with Presburger Arithmetic (BAPA) Constraints in Z3*. https://github.com/psuter/bapa-z3. 2012

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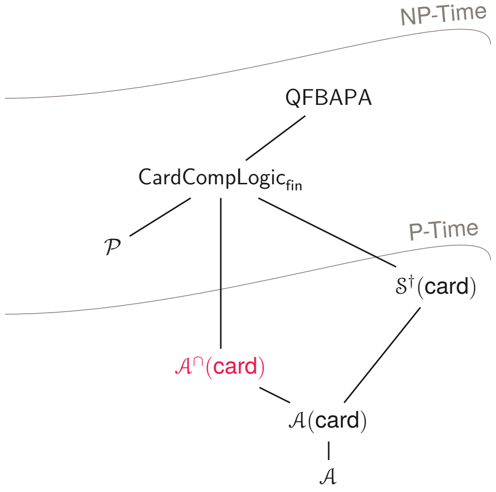

Boolean Algebra with
Presburger Arithmetic

Logic of Comparative Cardinality
(restricted to finite models)

$\mathcal{P}$: Propositional logic

$\mathcal{S}^\dagger$(card): $\mathcal{A}$(card) + $|a| > |b|$
+ Some + $\bar{a}$

$\mathcal{A}^\cap$(card): $\mathcal{A}$(card) + $\cap$

$\mathcal{A}$(card): $a \subseteq b$ + $|a| \geq |b|$

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I'm gonna conclude with this map, which shows exactly where $\mathcal{A}^\cap$(card) lies in the scheme of things. You can see $\mathcal{A}^\cap$(card) up there in red. At the top is Boolean Algebra with Presburger Arithmetic (well, without quantifiers), which probably some of you know. BAPA is much more expressive than our logic – it can perform inferences on a wide variety of both set and numerical terms, along with a cardinality map and propositional connectives $\wedge$, $\vee$, $\neg$. But remember that I began the talk with the fact that its inference is intractible. Actually, just the quantifier-free fragment of BAPA is NP-complete!

If we go a little less expressive, we land on this very recent Logic of Comparative Cardinality CardCompLogic. The full CardCompLogic has predicates for stating that a set is finite or infinite, but if we restrict their sets to be finite, we end up with a sublogic of QFBAPA. The jury is still out on the complexity of CardCompLogic, but with intersection, complement, and propositional connectives (so many ways to encode SAT!), I doubt that it has polynomial-time inference.

Another cousin of our logic is this syllogistic logic "S dagger card". This logic uses additional relations for strict cardinality comparison and Aristotle's "Some x are y", along with set complement rather than set intersection. This system manages to be polynomial-time decidable. We strongly suspect that if we were to have both set complement and set intersection, we would end up begin able to reduce the logic into SAT.

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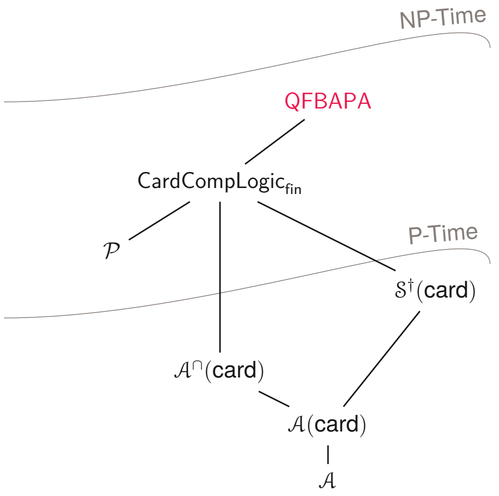

Boolean Algebra with Presburger Arithmetic

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Viktor Kuncak, Huu Hai Nguyen, and Martin Rinard. "Deciding Boolean algebra with Presburger arithmetic". In: *J. Automat. Reason.* 36.3 (2006), pp. 213–239. ISSN: 0168-7433. DOI: 10.1007/s10817-006-9042-1. URL: https://doi.org/10.1007/s10817-006-9042-1

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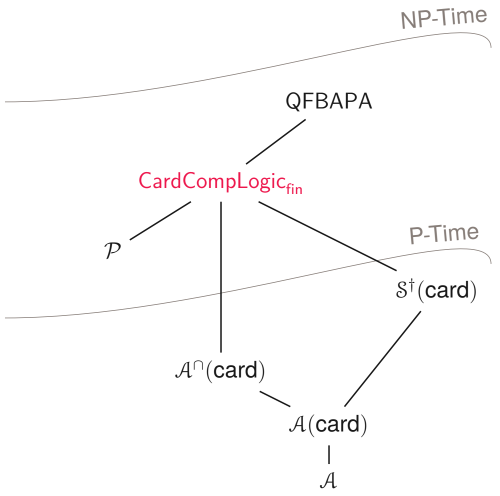

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Yifeng Ding, Matthew Harrison-Trainor, and Wesley Holliday. "The Logic of Comparative Cardinality". Retrieved from https://escholarship.org/uc/item/2nn3c35x (accessed on June 12, 2019). 2018

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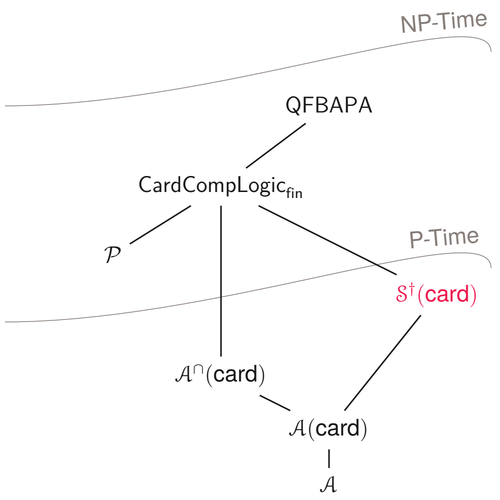

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Lawrence S. Moss. "Syllogistic logic with cardinality comparisons". In: *J. Michael Dunn on information based logics*. Vol. 8. Outst. Contrib. Log. Springer, 2016, pp. 391–415. DOI: 10.1007/978-3-319-29300-4_18. URL: https://doi.org/10.1007/978-3-319-29300-4_18

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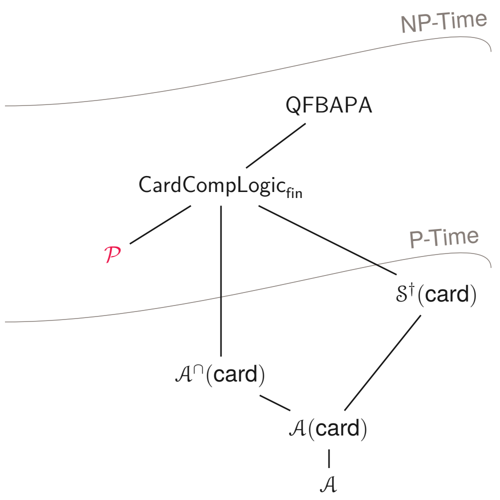

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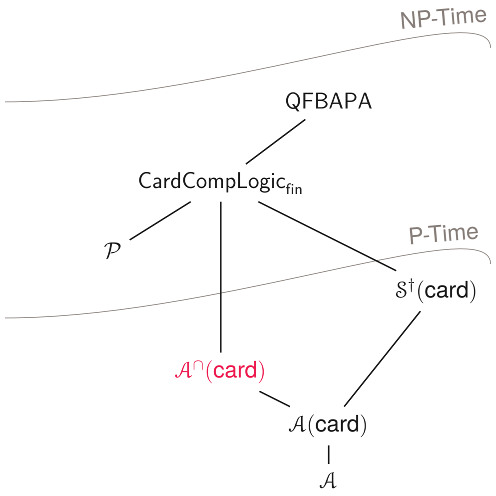

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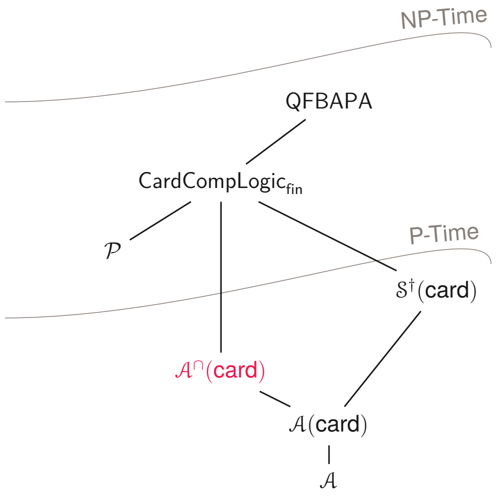
We're very interested in how much you can add to $\mathcal{A}^\cap$(card) while staying below the polytime line. We would love to get polynomial time with More. There is also the possibility that we would stay below if we integrate both union and intersection together. Another fun direction is to explore what kind of reasoning is polynomial time, but below propositional.

...

Thank You!

Questions?



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