Logics for Sizes with Union or Intersection

Caleb Kisby, Saúl A. Blanco,
Alex Kruckman, Lawrence S. Moss

What Kind of Inference?

$$\frac{\textit{All cats are mammals that purr} \quad \textit{There are at least as many cats as purring things}}{\textit{All purring things are cats}}$$

$$\frac{\textit{All hippies can vote} \quad \textit{All hippies are vegetarian} \quad \textit{All people can vote} \quad \textit{There are at least as many people as things that can vote} \quad \textit{There are at least as many hippies as people}}{\textit{All vegetarian people are hippies}}$$

Semantics

$[\![a]\!] \subseteq M$

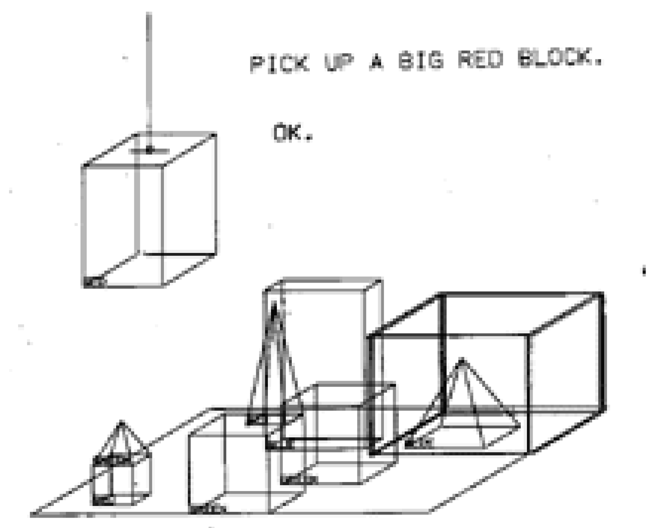
$[\![a \cap b]\!] = [\![a]\!] \cap [\![b]\!]$

$\mathcal{M} \models \mathsf{AtLeast}\ x\ y$ iff $|[\![x]\!]| \geq |[\![y]\!]|$

$\mathcal{M} \models \mathsf{All}\ x\ y$ iff $[\![x]\!] \subseteq [\![y]\!]$

$\Gamma \models \varphi$

Examples in AI



Inference Rules

$$\frac{}{\mathsf{All}\ x\ x}\ (\text{AXIOM}) \qquad \frac{\mathsf{All}\ x\ y \quad \mathsf{All}\ y\ z}{\mathsf{All}\ x\ z}\ (\text{BARBARA})$$

$$\frac{\mathsf{All}\ x\ y \quad \mathsf{AtLeast}\ x\ y}{\mathsf{All}\ y\ x}\ (\text{MIX}) \qquad \frac{\mathsf{All}\ x\ y}{\mathsf{AtLeast}\ y\ x}\ (\text{SIZE}) \qquad \frac{\mathsf{AtLeast}\ x\ y \quad \mathsf{AtLeast}\ y\ z}{\mathsf{AtLeast}\ x\ z}\ (\text{TRANS})$$

$$\frac{}{\mathsf{All}\ (a \cap b)\ a}\ (\text{INTER-L}) \qquad \frac{}{\mathsf{All}\ (a \cap b)\ b}\ (\text{INTER-R}) \qquad \frac{\mathsf{All}\ a\ b \quad \mathsf{All}\ a\ c}{\mathsf{All}\ a\ (b \cap c)}\ (\text{INTER-ALL})$$

$$\frac{}{\mathsf{All}\ a\ (a \cup b)}\ (\text{UNION-L}) \qquad \frac{}{\mathsf{All}\ b\ (a \cup b)}\ (\text{UNION-R}) \qquad \frac{\mathsf{All}\ a\ c \quad \mathsf{All}\ b\ c}{\mathsf{All}\ (a \cup b)\ c}\ (\text{UNION-ALL})$$

$$\frac{\mathsf{More}\ x\ y \quad \mathsf{AtLeast}\ y\ z}{\mathsf{More}\ x\ z}\ (\text{MORE-L}) \qquad \frac{\mathsf{AtLeast}\ x\ y \quad \mathsf{More}\ y\ z}{\mathsf{More}\ x\ z}\ (\text{MORE-R})$$

$$\frac{\mathsf{More}\ x\ y}{\mathsf{AtLeast}\ x\ y}\ (\text{MORE-ATLEAST}) \qquad \frac{\mathsf{More}\ z\ z}{\varphi}\ (\text{X}) \qquad \frac{\begin{matrix}[\mathsf{AtLeast}\ y\ x] \\ \vdots \\ \mathsf{More}\ z\ z\end{matrix}}{\mathsf{More}\ x\ y}\ (\text{RAA})$$

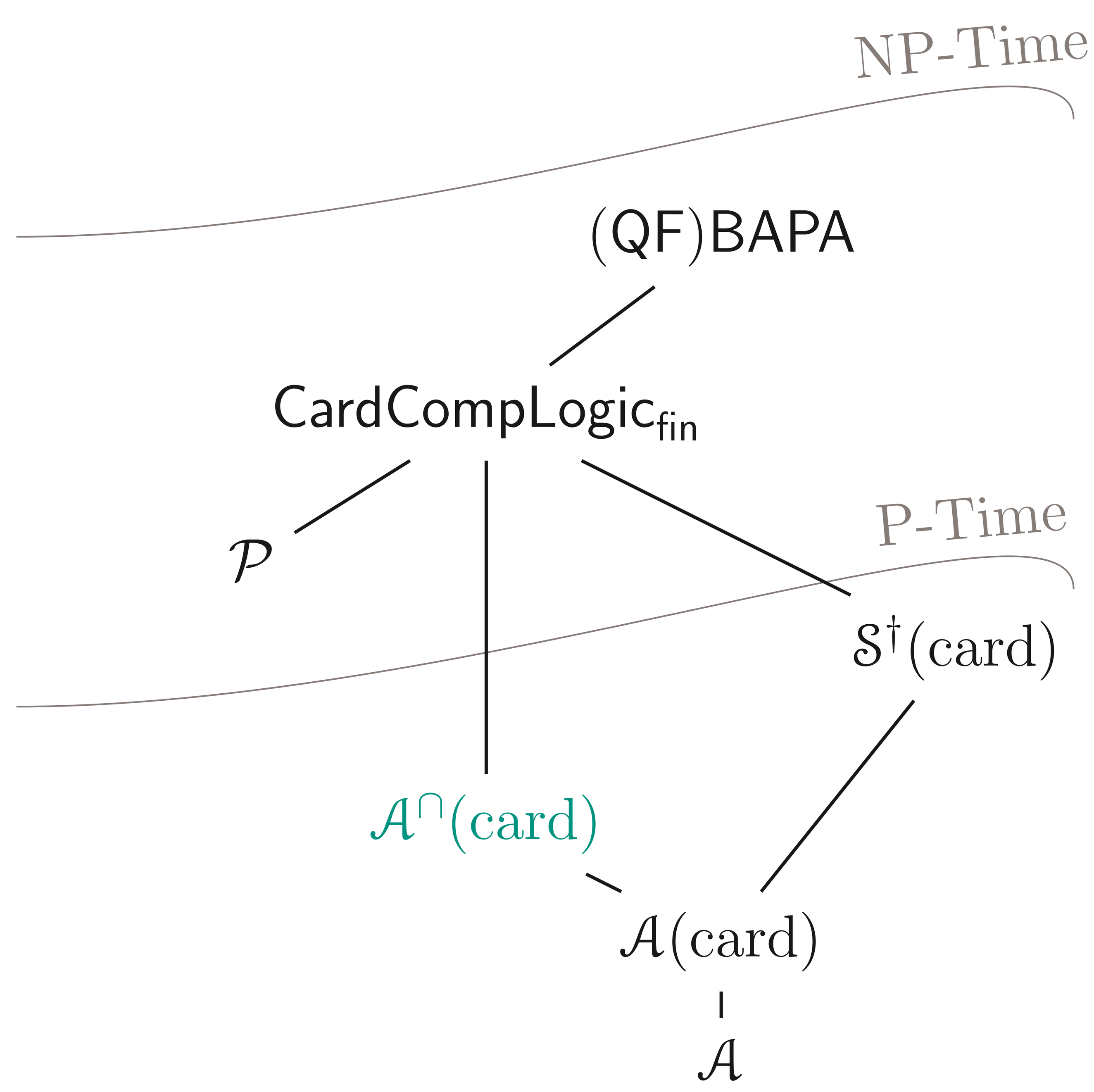
Main Technical Results

Theorem. *The logics* $\mathcal{A}^{\cap}(\text{card})$ *and* $\mathcal{A}^{\cup}(\text{card})$ *are complete.*

Theorem. $\vdash$ *is decidable in polynomial time!*

Theorem. *If* $\Gamma \nvdash \varphi$*, then we can construct a finite countermodel* $\mathcal{M}$ *satisfying* Γ *but falsifying* φ *in polynomial time*

The Map



Future Work

Polynomial Time with 'More'

$$\frac{\mathsf{All}\ x\ a \quad \mathsf{All}\ x\ b \quad \mathsf{More}\ (a \cup b)\ b}{\mathsf{More}\ a\ x}$$

Adding 'Some'

$$\frac{\mathsf{More}\ a\ b \quad \mathsf{AtLeast}\ c\ d \quad \mathsf{AtLeast}\ (b \cup d)\ (a \cup c)}{\mathsf{Some}\ a\ c}$$

Integrating with Z3 Plugin

- Our logic may best be viewed as an efficient fragment of (QF)BAPA
- Consequently, we keep an eye towards embedding $\mathcal{A}^{\cap}(\text{card})$ in an SMT solver
- Ideal candidate is *Z3* plugin [6] for BAPA

References

[1] Yifeng Ding et al. "The Logic of Comparative Cardinality". 2018.

[2] Viktor Kuncak and Martin Rinard. "Towards efficient satisfiability checking for boolean algebra with Presburger Arithmetic". 2007.

[3] Viktor Kuncak et al. "Deciding Boolean algebra with Presburger arithmetic". 2006.

[4] Lawrence S. Moss. "Syllogistic logic with cardinality comparisons". 2016.

[5] Ian Pratt-Hartmann. "On the Computational Complexity of the Numerically Definite Syllogistic and Related Logics". 2008.

[6] Philippe Suter et al. "Sets with Cardinality Constraints in Satisfiability Modulo Theories". 2011.